

## Assignment #5 – Influence

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**How to read this assignment :** Exercise levels are indicated as follows

( $\rightarrow$ ) “elementary”: the answer is not strictly speaking obvious, but it fits in a single sentence, and it is an immediate application of results covered in the lectures.

*Use them as a checkpoint: it is strongly advised to go back to your notes if the answer to one of these questions does not come to you in a few minutes.*

( $\curvearrowright$ ) “intermediary”: The answer to this question is not an immediate translation of results covered in class, it can be deduced from them with a reasonable effort.

*Use them as practice: how far are you from the answer? Do you still feel uncomfortable with some of the concepts and definitions? which part could you complete quickly?*

( $\nrightarrow$ ) “tortuous”: this question either requires an advanced concept, a proof that is long or inventive, or it is still open.

*Use them as an inspiration: can you answer any of them? does it bring you to another problem that you can answer or study further? It is recommended to work on this question only AFTER you are done with the rest!*

## PART A — MANIPULATING THE CONCEPTS

**Exercise 1: Game coordination and threshold model (8 pt)**

**Motivation:** This exercise allows you to bridge simple coordination games, also referred to as cooperative game, with threshold model seen in the course. A coordination game is any type of game where players chooses from the same set of strategies and turn out to have higher payoff when they make the same choice.

Assume that each player can choose either strategy A or B. Strategy A may for example denote the adoption of an innovation. We suppose that players are connected to each other according to a graph  $G = (V, E)$  where  $V$  denotes the set of all players and an edge  $(u, v)$  is in  $E$  if and only if two players are connected.

Every edge  $(u, v)$  corresponds to a game where players  $u$  and  $v$  are involved. In particular they all receive a pay-off for this game which only depends on which strategy each of this player has chosen according to the following table:

	$v$ chooses A	$v$ chooses B
$u$ chooses A	$a$	$0$
$u$ chooses B	$0$	$b$

where we assume that  $a$  and  $b$  are two real numbers such that  $a > b$ .

The total payoff receives by a user is the result of her payoff on all games she is involved in (*i.e.*, she is involved by one game for each incident edge).

1. ( $\rightarrow$ ) Given that all players except  $u$  has decided their strategy (which could either be A or B), and that  $u$  knows their choices, under which conditions will  $u$  rationally choose to play strategy  $A$  or  $B$ ?

We consider an infinite sequence of games indexed by time  $t = 0, 1, 2, \dots$ . We assume that at time  $t$  a user decides her strategy to maximize her payoff when all other players plays the same strategy as at time  $t - 1$  (which is her last observation).

2. ( $\rightarrow$ ) Let us assume that a set of players always choose to play strategy A independently of others, and that all other players initially play strategy B. How would you describe the evolution of this system with time  $t$ ? Can you compare it to one seen in the course?

## Exercise 2: Adoption with neighbor effect and renewed decision (7 pt)

**Motivation** Adoption of an innovation (like an online service) could be promoted by encouraging users to start the service for free. One can distinguish a permanent promotion (where users could access the service for free forever) and temporary promotion where they have a free period. Clearly the first form of promotion (which is the one we studied in class) can only do better. On the other hand, and especially for service paid by subscription, the second option seems cheaper overall to organize. The exercise answers the following question “Could this form of promotion be significantly less efficient?”

In this exercise, we propose to show that, according to macroscopic metric (*i.e.*, , the ability for a finite set of players to create an infinite cascade of adoption), the two are equivalent.

As in the previous exercise, we consider a set of users  $V$  that are connected together along edges of a graph  $G = (V, E)$ . We consider an infinite amount of time slots  $t = 0, 1, 2, \dots$ . We assume that all users have an adoption threshold  $\theta$  which characterizes their behavior as follows: during a time slot  $t$ , a user observes how many of her friends used the service and renew her subscription for time slot  $t + 1$  if only if at least a fraction  $\theta$  of her friend have used the service during time slot  $t$ .

In the *temporary promotion model* we consider an extreme version where initially a set of users  $S_0$  are proposed to use the service for free *for a single time slot*. At the end of this time slot, they may or may not renew the service depending on what they have observed and the threshold rule defined above, just like any other nodes in the network. The only effect of the promotion is to increase the set of users during the first time slot.

In the *targeted permanent promotion model* we assume that an initial set of users  $S_0$  are proposed the service for free *for an unlimited amount of time*, while all other users who may use the service or not decides to do so according to the threshold rule defined above.

In the *general permanent promotion model*, we assume that any user who decides to use the service once will receive a free subscription in all subsequent time slot. All users who have never used the service may decide to adopt it or not according to the threshold rule defined above.

Let us denote by  $S_t$  for  $t = 0, 1, \dots$  the set of users that decide to subscribe to the service during time slot  $t$  in the temporary promotion model. Let us define for each subset  $A \subseteq V$  the function  $f_\theta$  as

$$f_\theta(A) = \{ v \in V \mid \text{at least a fraction } q \geq \theta \text{ of neighbors of } v \text{ are in } A \} .$$

1. ( $\rightarrow$ ) Show that for any  $t \geq 0$ ,  $S_t = f_\theta^{(t)}(S_0)$  where  $f_\theta^{(t)}$  denotes the function  $f_\theta$  applied  $t$  times.

Let us denote respectively by  $S'_t$  and  $S''_t$  for  $t = 0, 1, \dots$  the set of users that decide to subscribe to the service during time slot  $t$  in the targeted permanent promotion model and the general permanent promotion model.

2. ( $\curvearrowright$ ) Show that  $S'_t = S''_t = g_\theta^{(t)}(S_0)$  where for any subset  $A \subseteq V$ ,  $g_\theta(A) = f_\theta(A) \cup A$ . What can you deduce w.r.t. the efficiency of the general permanent promotion strategy?

We consider an infinite graph  $G = (V, E)$  where all nodes have finite degree. For a given threshold  $\theta$ , we say that a set  $S_0$  is an infectious set for the temporary promotion model if, starting from  $S_0$  the sequence  $S_t$  of nodes that subscribe the service eventually reach all nodes. Formally,  $S_0$  is infectious if

$$\forall v \in V, \exists k \geq 0 \text{ such that } \forall t \geq k, v \in S_t .$$

Note that this definition is shown here for the temporary promotion model (*i.e.*, using the sequence  $(S_t)_{t \geq 0}$ ) and that the same definition could be used with  $S'_t$  and  $S''_t$  to define infectious set in the two other models.

3. ( $\curvearrowright$ ) Give an example of a graph  $G = (V, E)$  and a set  $S_0$  that is infectious for the general permanent promotion model but not for the temporary promotion model.

Let  $S_0$  be a finite subset that is infectious for the general permanent promotion model. We define  $S^+$  as the subset that contains all nodes in  $S_0$  as well as all neighbors of nodes in  $S_0$ . Since  $S_0$  is infectious for the general permanent promotion model, there exists  $t_0$  such that  $S^+ \subseteq S'_{t_0}$ , where  $S'_t$  is the sequence of subscribing nodes starting from  $S_0$ .

4. ( $\curvearrowright$ ) Show that the subset  $T = S'_{t_0}$  is infectious for the temporary promotion model (*i.e.*, that the sequence  $(S_t)_{t \geq 0}$  starting from T eventually contains the whole set).

### Exercise 3: Connection between two general models of influence (5.5 + 0.5( $\curvearrowright$ )pt)

**Motivation** In this exercise, we prove that the two general models of influence with random thresholds are, under a natural condition, equivalent.

As a quick reminder from the lecture, one can define more general model of influence in one of the two following manner:

- Define for any node  $u \in V$  a function  $g_u$  taking value in  $[0; 1]$  and which is defined on all subset of neighbors of  $u$  (*i.e.*, for any  $S \subseteq N(u)$  we define a value  $g_u(S) \in [0; 1]$ ).

Node's behavior is then characterized as follows. First, we assume that a set  $S_0$  of nodes initially adopt the service, and that for any node  $v \in V$  there exists a threshold  $\theta_v$ , which chosen once for all in  $[0; 1]$  according to a uniform distribution.

Then, for any time slot  $t$ , if during this time slot  $t$ , the set of neighbors of  $v$  which adopt the service is  $S$ ,  $v$  will adopt the service if and only if we have  $\theta_v \leq g_v(S)$ .

- In another model, we assume that for any  $u$  and  $v$  such that  $(u, v)$  is an edge in  $E$  there exists a function  $p_v(u, \cdot)$  which takes value in  $[0; 1]$  and is defined on all subset of neighbors of  $v$  that do not contain  $u$  (*i.e.*, for any  $S \subseteq N(v)$  such that  $u \notin S$ , we define a value  $p_v(u, S)$ ).

The behavior of the nodes is then described as follows. A subset  $S_0$  of nodes initially adopt the service or the innovation at time  $t = 0$ .

Then for any  $t = 0, 1, \dots$ , whenever a node  $u$  adopts the service for the first time during  $t$ , for any node  $v$  that is a neighbor of  $u$  it makes a single attempt during this time slot to influence  $v$ . Note that if  $v$  has already been influenced and use the service nothing will happen. Otherwise it

indicates that all previous attempt to influence  $v$  has failed. We denote by  $S_v$  the set of all nodes who attempted to influence  $v$  before  $u$ . What happens then is the following. With probability  $p_v(u, S_v)$  (chosen independently from the past) the attempt is successful and  $v$  starts using the service at time  $t + 1$ . Otherwise, and hence with a probability  $(1 - p_v(u, S_v))$  the attempt does not succeed and hence  $u$  is added to  $S_v$ .

1. ( $\rightarrow$ ) The second model (using function  $p_v$ ) is called order independent if, for a node  $v$ , the probability that it adopts the service after an attempt by nodes  $u_1, u_2, \dots, u_k$  does not depend on the order of the sequence but only on the set  $\{u_1, \dots, u_k\}$ .

Write the probability that  $v$  adopts the service if a set  $S$  of nodes attempt to influence  $v$ . Provide an example where this probability is not order independent.

2. ( $\curvearrowright$ ) We assume that function  $g_v$  are all monotone (*i.e.*,  $g_v(S) \leq g_v(T)$  when  $S \subseteq T$ ). Show that the dynamics of adoption defined by  $g$  is equivalent to that defined by  $p_v$  if  $p_v$  satisfies:

$$p_v(u, S) = \frac{g_v(S \cup \{u\}) - g_v(S)}{1 - g_v(S)}.$$

3. ( $\curvearrowleft$ ) Similarly, show that if the functions  $p_v$  for all  $v$  follow the order independent property, the dynamics defined by  $p_v$  is equivalent to one defined using  $g_v$  for a proper choice of  $g_v$ .
4. ( $\leftrightarrow$ ) Note that  $g_v$  and  $p_v(u, \cdot)$  for any  $v$  and any  $u$  are real valued functions defined on set. Hence, they may or may not be submodular (according to the definition seen in the class), and it makes sense to say that the first model (resp. the second) is submodular if and only if all functions  $g_v$  (resp. all functions  $p_v(u, \cdot)$ ) are submodular.

We have just seen that the two models are equivalent in the sense that for any model defined with  $(g_v)_{v \in V}$  there is an equivalent dynamics that can be defined using  $(p_v(u, \cdot))_{u, v \in V}$ . So it does not really matter which definitions is used.

Does this imply that any dynamics that is submodular for the first model is also submodular for the second model? Provide either a proof or a counterexample for each inclusion.